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On the twisted Daehee polynomials with q -parameter

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Abstract

The n th twisted Daehee numbers with q -parameter are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind. In this paper, we give a p -adic integral representation of the twisted Daehee polynomials with q -parameter, and we derive some interesting properties related to the n th twisted Daehee polynomials with q -parameter.

Keywords: Bernoulli polynomials; Daehee polynomials with q -parameter; p -adic invariant integral

1 Introduction

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completions of algebraic closure of \mathbb{Q}_p . The p -adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on \mathbb{Z}_p is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x) \quad (\text{see [1-3]}). \quad (1.1)$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}. \quad (1.2)$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (1.3)$$

and the *Stirling number of the second kind* is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} \quad (1.4)$$

(see [4–6]).

Unsigned Stirling numbers of the first kind are given by

$$x^n = x(x+1) \cdots (x+n-1) = \sum_{l=0}^n |S_1(n, l)| x^l. \quad (1.5)$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned} (-x)_n &= (-1)^n x^n = \sum_{l=0}^n S_1(n, l) (-1)^l x^l \\ &= (-1)^n \sum_{l=0}^n |S_1(n, l)| x^l. \end{aligned} \quad (1.6)$$

Hence $S_1(n, l) = |S_1(n, l)| (-1)^{n-l}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order r* are defined by the generating function to be

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1, 4, 7–18]}). \quad (1.7)$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order r* , and in the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the *ordinary Bernoulli polynomials*.

For $n \in \mathbb{N}$, let T_p be the p -adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \rightarrow \infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n .

We assume that q is an indeterminate in \mathbb{C}_p with $|1 - q|_p < p^{-\frac{1}{p-1}}$. Then we define the q -analog of a falling factorial sequence as follows:

$$(x)_{n,q} = x(x-q)(x-2q) \cdots (x-(n-1)q) \quad (n \geq 1), \quad (x)_{0,q} = 1.$$

Note that

$$\lim_{q \rightarrow 1} (x)_{n,q} = (x)_n = \sum_{l=0}^n S_1(n, l) x^l.$$

Recently, DS Kim and T Kim introduced the *Daehee polynomials* as follows:

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y) \quad (n \geq 0) \text{ (see [2, 9, 19]).} \quad (1.8)$$

When $x = 0$, $D_n = D_n(0)$ are called the *n th Daehee numbers*. From (1.8), we can derive the generating function to be

$$\left(\frac{\log(1+t)}{t} \right) (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} \quad (\text{see [9]}). \quad (1.9)$$

In addition, DS Kim *et al.* consider the *Daehee polynomials with q -parameter*, which are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n,q} \frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\log(1+qt)}{q((1+qt)^{\frac{1}{q}} - 1)} \quad (\text{see [20, 21]}). \quad (1.10)$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the *Daehee numbers with q -parameter*.

From the viewpoint of a generalization of the Daehee polynomials with q -parameter, we consider the *twisted Daehee polynomials with q -parameter*, defined to be

$$\sum_{n=0}^{\infty} D_{n,\xi,q} \frac{t^n}{n!} = (1+q\xi t)^{\frac{x}{q}} \frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}} - 1)}, \quad (1.11)$$

where $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$ and $\xi \in T_p$.

In this paper, we give a p -adic integral representation of the twisted Daehee polynomials with q -parameter, which is called the Witt-type formula for the twisted Daehee polynomials with q -parameter. We can derive some interesting properties related to the n th twisted Daehee polynomials with q -parameter.

2 Witt-type formula for the n th twisted Daehee polynomials with q -parameter

First, we consider the following integral representation associated with falling factorial sequences:

$$\xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} d\mu_0(y), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \text{ and } \xi \in T_p. \quad (2.1)$$

By (2.1),

$$\begin{aligned} \sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} d\mu_0(y) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{x+y}{q} \right)_n d\mu_0(y) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x+y}{q}} d\mu_0(y), \end{aligned} \quad (2.2)$$

where $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, put $f(x) = (1 + q\xi t)^{\frac{x+y}{q}}$. By (1.1), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} (1 + q\xi t)^{\frac{x+y}{q}} d\mu_0(y) &= (1 + q\xi t)^{\frac{x}{q}} \frac{\log(1 + q\xi t)}{q((1 + q\xi t)^{\frac{1}{q}} - 1)} \\ &= \sum_{n=0}^{\infty} D_{n,\xi,q}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.3)$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1 For $n \geq 0$, we have

$$D_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (x + y)_{n,q} d\mu_0(y).$$

In (2.3), by replacing t by $\frac{1}{\xi q}(e^{\xi t} - 1)$, we have

$$\sum_{n=0}^{\infty} D_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \frac{(e^{\xi t} - 1)^n}{n!} = e^{\frac{\xi t x}{q}} \frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}} - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{\xi^n}{q^n} \frac{t^n}{n!} \quad (2.4)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \frac{1}{n!} (e^{\xi t} - 1)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^m \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \xi^m S_2(m, n) \frac{t^m}{m!}. \end{aligned} \quad (2.5)$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2 For $n \geq 0$, we have

$$B_n(x) = \sum_{m=0}^n D_{m,\xi,q}(x) \xi^{-m} q^{n-m} S_2(n, m).$$

By Theorem 2.1,

$$\begin{aligned} D_{n,\xi,q}(x) &= \xi^n \int_{\mathbb{Z}_p} (x + y)_{n,q} d\mu_0(y) \\ &= \xi^n q^n \sum_{l=0}^n \frac{1}{q^l} S_1(n, l) \int_{\mathbb{Z}_p} (x + y)^l d\mu_0(y). \end{aligned} \quad (2.6)$$

By (1.2), we can derive easily that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) &= \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} (x + y)^l d\mu_0(y) \frac{t^l}{l!}, \end{aligned} \quad (2.7)$$

and so

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y). \quad (2.8)$$

By (1.6), (2.7), and (2.8), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$D_{n,\xi,q}(x) = \xi^n \sum_{l=0}^n q^{n-l} S_1(n,l) B_l(x) = \xi^n \sum_{l=0}^n |S_1(n,l)| (-q)^{n-l} B_l(x).$$

From now on, we consider *twisted Daehee polynomials of order $k \in \mathbb{N}$ with q -parameter*. Twisted Daehee polynomials of order $k \in \mathbb{N}$ with q -parameter are defined by the multi-variant p -adic invariant integral on \mathbb{Z}_p :

$$D_{n,\xi,q}^{(k)}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_{n,q} d\mu_0(x_1) \cdots d\mu_0(x_k), \quad (2.9)$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{n,\xi,q}^{(k)} = D_{n,\xi,q}^{(k)}(0)$ are called the *Daehee numbers of order k with q -parameter*.

From (2.9), we can derive the generating function of $D_{n,\xi,q}^{(k)}(x)$ as follows:

$$\begin{aligned} & \sum_{n=0}^{\infty} D_{n,\xi,q}^{(k)}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1 + \cdots + x_k + x}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k) t^n \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + q\xi t)^{\frac{x_1 + \cdots + x_k + x}{q}} d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= (1 + q\xi t)^{\frac{x}{q}} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + q\xi t)^{\frac{x_1 + \cdots + x_k}{q}} d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= (1 + q\xi t)^{\frac{x}{q}} \left(\frac{\log(1 + q\xi t)}{q((1 + q\xi t)^{\frac{1}{q}} - 1)} \right)^k. \end{aligned} \quad (2.10)$$

Note that, by (2.9),

$$D_{n,\xi,q}^{(k)}(x) = \xi^n q^n \sum_{m=0}^n \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^m d\mu_0(x_1) \cdots d\mu_0(x_k). \quad (2.11)$$

Since

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1 + \cdots + x_k + x)t} d\mu_0(x_1) \cdots d\mu_0(x_k) = \left(\frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},$$

we can derive easily

$$B_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^n d\mu_0(x_1) \cdots d\mu_0(x_k). \quad (2.12)$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned} D_{n,\xi,q}^{(k)}(x) &= \xi^n q^n \sum_{m=0}^n \frac{S_1(n,m)}{q^m} B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^n q^{n-m} S_1(n,m) B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^n |S_1(n,m)| (-q)^{n-m} B_m^{(k)}(x). \end{aligned} \quad (2.13)$$

In (2.10), by replacing t by $\frac{1}{q\xi}(e^{\xi t} - 1)$, we get

$$\sum_{n=0}^{\infty} D_{n,\xi,q}^{(k)}(x) \frac{(e^{\xi t} - 1)^n}{\xi^n q^n n!} = e^{\frac{\xi t x}{q}} \left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}} - 1} \right)^k = \sum_{n=0}^{\infty} \frac{\xi^n B_n^{(k)}(x)}{q^n} \frac{t^n}{n!} \quad (2.14)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \frac{1}{n!} (e^{\xi t} - 1)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l,n) \xi^l \frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty} \left(\xi^m \sum_{n=0}^m \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} S_2(m,n) \right) \frac{t^m}{m!}. \end{aligned} \quad (2.15)$$

By (2.13), (2.14), and (2.15), we obtain the following theorem.

Theorem 2.4 For $n \geq 0$ and $k \in \mathbb{N}$, we have

$$D_{n,\xi,q}^{(k)}(x) = \xi^n \sum_{m=0}^n q^{n-m} S_1(n,m) B_m^{(k)}(x) = \xi^n \sum_{m=0}^n |S_1(n,m)| (-q)^{n-m} B_m^{(k)}(x)$$

and

$$B_n^{(k)}(x) = \sum_{m=0}^n D_{m,\xi,q}^{(k)}(x) \xi^{-m} q^{n-m} S_2(n,m).$$

Now, we consider the *twisted Daehee polynomials of the second kind with q -parameter* as follows:

$$\hat{D}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (-y + x)_{n,q} d\mu_0(y) \quad (n \geq 0). \quad (2.16)$$

In the special case $x = 0$, $\hat{D}_{n,\xi,q}(0) = \hat{D}_{n,\xi,q}$ are called the *twisted Daehee numbers of the second kind with q -parameter*.

By (2.16), we have

$$\hat{D}_{n,\xi,q}(x) = \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{-y + x}{q} \right)_n d\mu_0(y), \quad (2.17)$$

and so we can derive the generating function of $\hat{D}_{n,\xi,q}(x)$ by (1.1) as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \hat{D}_{n,\xi,q}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} q^n \xi^n \int_{\mathbb{Z}_p} \left(\frac{-y+x}{q} \right)_n d\mu_0(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} q^n \xi^n \int_{\mathbb{Z}_p} \left(\frac{-y+x}{q} \right)_n d\mu_0(y) t^n \\ &= \int_{\mathbb{Z}_p} (1 + q\xi t)^{\frac{-y+x}{q}} d\mu_0(y) \\ &= (1 + q\xi t)^{\frac{x}{q}} \frac{\log(1 + q\xi t)}{q((1 + q\xi t)^{\frac{1}{q}} - 1)} (1 + q\xi t)^{\frac{1}{q}}. \end{aligned} \quad (2.18)$$

From (1.3), (1.6), and (2.17), we get

$$\begin{aligned} \hat{D}_{n,\xi,q}(x) &= q^n \xi^n \int_{\mathbb{Z}_p} \left(\frac{-y+x}{q} \right)_n d\mu_0(y) \\ &= q^n \xi^n \int_{\mathbb{Z}_p} \sum_{l=0}^n \frac{S_1(n,l)}{q^l} (-y+x)^l d\mu_0(y) \\ &= \xi^n \sum_{l=0}^n S_1(n,l) (-1)^l \int_{\mathbb{Z}_p} (y-x)^l d\mu_0(y) q^{n-l} \\ &= \xi^n \sum_{l=0}^n S_1(n,l) (-1)^l B_l(-x) q^{n-l} \\ &= (-\xi)^n \sum_{l=0}^n |S_1(n,l)| B_l(-x) q^{n-l}. \end{aligned} \quad (2.19)$$

By (1.10), it is easy to show that $B_n(-x) = (-1)^n B_n(x+1)$. Thus, from (2.19), we have the following theorem.

Theorem 2.5 For $n \geq 0$, we have

$$\hat{D}_{n,\xi,q}(x) = \xi^n \sum_{l=0}^n S_1(n,l) (-1)^l B_l(-x) q^{n-l} = \xi^n \sum_{l=0}^n |S_1(n,l)| B_l(x+1) (-q)^{n-l}.$$

By replacing t by $\frac{1}{q\xi}(e^{\xi t} - 1)$ in (2.18), we have

$$\sum_{n=0}^{\infty} \hat{D}_{n,\xi,q}(x) \frac{1}{q^n \xi^n} \frac{(e^{\xi t} - 1)^n}{n!} = e^{\frac{\xi t}{q}(x+1)} \frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}} - 1} = \sum_{n=0}^{\infty} \frac{\xi^n B_n(x+1)}{q^n} \frac{t^n}{n!} \quad (2.20)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \hat{D}_{n,\xi,q}(x) \frac{1}{q^n \xi^n} \frac{(e^{\xi t} - 1)^n}{n!} &= \sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi,q}(x)}{q^n \xi^n} \sum_{m=n}^{\infty} S_2(m,n) \frac{(\xi t)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \hat{D}_{m,\xi,q}(x) S_2(n,m) q^{-m} \xi^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.21)$$

By (2.20) and (2.21), we obtain the following theorem.

Theorem 2.6 For $n \geq 0$, we have

$$B_n(x+1) = \sum_{m=0}^n q^{n-m} \xi^{-m} \hat{D}_{m,\xi,q}(x) S_2(n,m).$$

Now, we consider *higher-order twisted Daehee polynomials of the second kind with q -parameter*. Higher-order twisted Daehee polynomials of the second kind with q -parameter are defined by the multivariate p -adic invariant integral on \mathbb{Z}_p :

$$\hat{D}_{n,\xi,q}^{(k)}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_{n,q} d\mu_0(x_1) \cdots d\mu_0(x_k), \quad (2.22)$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $\hat{D}_{n,\xi,q}^{(k)} = \hat{D}_{n,\xi,q}^{(k)}(0)$ are called the *higher-order twisted Daehee numbers of the second kind with q -parameter*.

From (2.22), we can derive the generating function of $\hat{D}_{n,\xi,q}^{(k)}(x)$ as follows:

$$\begin{aligned} & \sum_{n=0}^{\infty} \hat{D}_{n,\xi,q}^{(k)}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{-x_1 - \cdots - x_k + x}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k) t^n \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + q\xi t)^{\frac{-x_1 - \cdots - x_k + x}{q}} d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= (1 + q\xi t)^{\frac{x+k}{q}} \left(\frac{\log(1 + q\xi t)}{q((1 + q\xi t)^{\frac{1}{q}} - 1)} \right)^k. \end{aligned} \quad (2.23)$$

By (2.22),

$$\begin{aligned} & \hat{D}_{n,\xi,q}^{(k)}(x) \\ &= \xi^n q^n \sum_{m=0}^n \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)^m d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= \xi^n q^n \sum_{m=0}^n \frac{S_1(n,m)}{(-q)^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^m d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= \xi^n q^n \sum_{m=0}^n \frac{S_1(n,m)}{(-q)^m} B_m^{(k)}(-x) \\ &= \xi^n \sum_{m=0}^n q^{n-m} |S_1(n,m)| B_m^{(k)}(-x). \end{aligned} \quad (2.24)$$

From (1.10), we know that $B_n^{(k)}(-x) = (-1)^n B_n^{(k)}(k+x)$. Hence, by (2.24), we obtain the following theorem.

Theorem 2.7 For $n \geq 0$, we have

$$\hat{D}_{n,\xi,q}^{(k)}(x) = \xi^n \sum_{m=0}^n (-1)^m q^{n-m} S_1(n,m) B_m^{(k)}(-x) = \xi^n \sum_{m=0}^n (-1)^m q^{n-m} |S_1(n,m)| B_m^{(k)}(x+k).$$

In (2.23), by replacing t by $\frac{1}{q\xi}(e^{\xi t} - 1)$, we get

$$\sum_{n=0}^{\infty} \hat{D}_{n,\xi,q}^{(k)}(x) \frac{(e^{\xi t} - 1)^n}{\xi^n q^n n!} = e^{\frac{\xi t}{q}(x+k)} \left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}} - 1} \right)^k = \sum_{n=0}^{\infty} \frac{\xi^n B_n^{(k)}(x+k)}{q^n} \frac{t^n}{n!} \quad (2.25)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \frac{1}{n!} (e^{\xi t} - 1)^n &= \sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l, n) \xi^l \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\xi^n \sum_{m=0}^n \frac{\hat{D}_{m,\xi,q}^{(k)}(x)}{\xi^m q^m} S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.26)$$

By (2.25) and (2.26), we obtain the following theorem.

Theorem 2.8 For $n \geq 0$ and $k \in \mathbb{N}$, we have

$$B_n^{(k)}(x+k) = \sum_{m=0}^n \hat{D}_{m,\xi,q}^{(k)}(x) \xi^{-m} q^{n-m} S_2(n, m).$$

Competing interests

The author declares that they have no competing interests.

Author's contributions

The author contributed to the manuscript and typed, read, and approved the final manuscript.

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